Synthetic seismograms for body waves: an overview*
Harold M. Mooney†

As a starting point, we will consider a seismologist and a seismogram. The seismologist may represent some speciality such as petroleum exploration, earthquake seismology or civil engineering. The seismogram in its simplest form will provide a graph of ground motion versus time (a 'time history'). The seismogram should be accompanied by information which enables the seismologist to relate it back to true ground motion. Typically, the ground motion itself has been converted into an electrical signal by a transducer (exploration geophone, hydrophone, earthquake seismometer, accelerometer), following which the electrical signal has been converted to a visual display.

The seismologist will always embark upon the interpretation problem with some kind of geological or mechanical model in mind. The model may be a very simple one or it may represent some complex geological condition, but it will necessarily be much simpler than the true natural condition. In most cases, the model underlying the seismogram can be conveniently broken down into three parts: the source, the transmitting medium and the receiver (Fig. 1).

The seismologist's ultimate goal will be to infer information, previously unknown, about one of the three components:

(1) The medium. Most commonly, the seismologist wishes to learn something about the medium. A petroleum seismologist may wish to infer depths to geological interfaces' or porosity and fluid content of selected layers, or to detect the presence of oil and gas. An earthquake seismologist may wish to map geological interfaces in the Earth's deep interior, or to characterise the subsurface materials in terms of seismic velocity, density, and chemical and petrological properties.

(2) The source. A petroleum seismologist may wish to establish the character of the seismic waveform. An earthquake seismologist may wish to infer the mechanical properties of the earthquake source, such as orientation and direction of motion on the fault, or the mechanical model which best describes the earthquake source.

(3) The receiver. Whilst information on the receiver is least commonly sought (since alternative laboratory means may be available by which to obtain it), the receiver–ground coupling problem can be investigated in this way (Stagg and Engler 1980).

The seismologist has now defined goals and possesses one or more seismograms. The 'interpretation' of these seismograms will, we hope, contribute to achieving these goals.

The classic model-building approach to geophysical interpretation proceeds as follows: (a) use the model to predict what the observations should have been and (b) modify the model on a trial-and-error basis to improve agreement between prediction and observation. Synthetic seismograms belong to step (a). The entire process may be described as a 'forward-modelling' method of interpretation. An 'inverse-modelling' method would systematize and perhaps automate step (b) to achieve the same result. The applications of synthetic seismograms to date have been largely confined to forward modelling.

Interpretation by these methods will always be ambiguous. This means that additional solutions (other geological structures, for example) may exist which can account for the observations equally well within the limits of accuracy of the data. These alternative solutions may differ greatly from the first solution. Synthetic seismograms provide a powerful method to narrow the range of acceptable models and thereby reduce the severity of the non-uniqueness problem. They achieve this by making such detailed predictions of seismic waveform (amplitude, phase, arrival time) that very few models can be found which fit the observations.

We may cite three major uses for synthetic seismograms (Pilant 1979):

(1) Improvement or refinement of a geological model. The amplitude and phase information from a synthetic seismogram can greatly narrow the range of models fitting travel-time interpretation. An example would be the work of earthquake seismologists in refining the velocity structure of the crust and mantle.
(2) Determination of relative importance of seismic phases. An example arises in reflection seismic exploration. With hundreds of reflecting interfaces and thousands of possible ray paths including multiples, we normally encounter difficulty in identifying the ray paths associated with the few observed reflections. A synthetic seismogram can yield relative amplitudes for the thousands of possible ray paths and thereby identify those with large predicted amplitudes.

(3) Use of a partial model to determine what portion of an observed seismogram it can account for. We may encounter a geological model which includes complications (edges, corners, etc) that we cannot handle by synthetic seismogram comparisons. We can, however, compute synthetic seismograms for a portion of the model as, for example, for layering. Thus, we may be able to account for some of the observed effects, leaving the remainder for different treatment. For example, Hatherly (1983) subtracts synthetic uniform-medium waveforms from layered-medium waveforms the better to display head waves and reflections.

We may note some fields of application. In petroleum geophysics, synthetic seismograms have been used as a link between observed seismograms and well logs. The well logs provide the data for the geological model (velocity, density, etc). The synthetics can be computed from such data and compared with the observations. Since the ray path for the synthetic can usually be determined unambiguously, the ray path on the observed seismogram can be inferred by correlation. A more recent technique takes a pseudo-inverse approach, in which the observed seismogram is converted to a synthetic well log.

Earthquake seismologists and to a lesser extent seismologists using explosive sources to study crustal structure have used synthetic seismograms. One major goal has been to characterise the earthquake source: the displacement history, the force distribution, etc. Some remarkable agreements have been obtained; an example is shown later. The fundamental non-uniqueness has not been removed, but the agreement has been obtained to such fine detail that the credibility of the solution is high.

Synthetic seismograms have not found much use for engineering and mining seismic applications, but the techniques are available and could be adapted. Part of the lack of application arises from the simplified subsurface models which are acceptable in these fields (uniform layers, dipping plane interfaces, etc). We will show an example later, however, which displays the striking waveform change to be expected in wide-angle shallow reflection seismology.

**Specification of the Model. I: Introduction**

A geologist will recognise and indeed insist that the complexity of the Earth as a medium for propagating seismic waves is beyond human comprehension. As we progress down the scale of size from the whole Earth to an ocean basin to a mountain range to an ore body to a mineral grain, we find at each stage that the actual structure or object is too complicated to comprehend, much less to describe fully.

A major part of the geologist's discipline is to simplify objects to the point where they can be described. This inevitably produces conflicting requirements. On the one hand, the model has to be simplified to the point where description is not only possible but usable. On the other hand, if the simplification is carried too far, the model may fail to represent the significant features of the real world.

The geologist may wish simply to describe the materials including their physical and chemical properties. His/her purpose is to understand their history and to predict their behaviour and relationships at inaccessible depths. The geophysicist, including especially the seismologist, must push the simplification even further than the geologist. A model is required of sufficient simplicity that the working of physical laws upon it can be treated mathematically. In seismology, we need to be able to apply the theory of the mechanics of deformable bodies. Such mathematics have been highly developed for uniform bodies, but as soon as the body becomes non-uniform in any significant way, the mathematics become more difficult and even intractable.

In following sections, we first consider Earth models at varying levels of complexity, noting the conflict between improved models and more difficult analysis. The model must include not only the Earth medium itself but the source and the receiver. Secondly, we consider the mathematics of seismic wave propagation. An exact solution to these equations will be possible only for the very simplest Earth models. Such solutions are useful, of course, and must not be neglected. They provide insight into the seismic process, they serve as controls, checks and starting points for more complex models, and they may sometimes provide all of the information needed.

Where exact solutions are not available we may consider two approaches. In the first, we seek approximate solutions to the mathematical problem, valid with limited accuracy or under restricted conditions. In the second, we use the computer to obtain numerical solutions. The latter approach may yield excellent results but often fails to provide insight into a class of solutions.

Current research in synthetic seismogram computations is largely directed toward a combination of the above approaches. We seek first to obtain approximate solutions to the equations for geological models of moderate but increasing complexity. The various kinds of approximation lead to various methods with names such as Asymptotic Ray Theory, Generalised Ray
Theory, Reflectivity Method, etc. Each of the methods has found use for certain classes of problems. Numerical solutions by computer then provide the desired synthetic seismograms.

Specification of the Model. II: The Earth Model
The specification of an Earth model describes two aspects of the model: dimensions and shape and physical properties. Neither of these can ever be an exact representation of the real Earth, so approximations of varying degrees of severity must be made. We will describe some of these approximations:

Geometry of the free surface
The free surface is a boundary with special importance because it establishes the limit beyond which seismic waves cannot propagate. Further, the receiver and the source are usually (although not always) located on the free surface.

Three approximations are in common use:
(1) A plane surface. The so-called ‘flat Earth approximation’ includes this assumption. It represents a valid approximation even on a spherical earth up to a source–receiver distance of several hundred km. Thus, it is widely used in petroleum exploration, engineering seismology, crustal studies and near-earthquake studies. We note that the usefulness of this approximation is increased by a mathematical artifice called the ‘Earth-flattening approximation’ in which the mathematics of a spherically symmetrical earth are transformed to those of an equivalent flat Earth.
(2) A spherical surface. This approximation is widely used in earthquake seismology.
(3) The actual (or near-actual) topography over a limited area. This approximation finds use in petroleum seismology and in engineering seismology, usually in the form of an elevation correction.

Geometry and location of internal boundaries, if any
A boundary is usually defined as a first (or second) order discontinuity in one or more of the physical properties listed below.

Physical properties of the model
(1) Listing of the properties of the model: \( V_p \), compressional velocity; \( V_s \), shear velocity; \( D \), density; \( Q \), anelastic attenuating property.
(2) Spatial variation of the physical properties.

The geometry and location of internal boundaries and the physical properties of the model are intertwined. It is at the point of specifying these quantities that the conflict becomes most apparent between realistic representation of the Earth on the one hand and mathematical tractability on the other.

The anelastic attenuating property presumably originates in frictional heat and other losses during passage of the seismic wave. The symbol \( Q \) for quality factor was originally introduced by electrical engineers to describe circuit losses. In seismology, the reciprocal \( Q^{-1} \) correlates with increasing attenuation. Another physical property capable of inclusion in synthetic seismogram calculations is seismic anisotropy such as occurs in a schist or a laminated sedimentary rock. This has been taken into account by Booth and Crampin (1983).

The simplest geological model consists of uniform material everywhere beneath the free surface. Mathematical solutions are relatively easy to obtain. These solutions have value in providing physical insight and in supplying checks (via limiting cases) on more complicated structures. The next simplest class of models includes those in which velocity and density vary with only one coordinate, usually depth. In these models, we have, for example, \( V_p = V_p(z) \) if \( z \) is depth beneath the free surface in the flat Earth case, or \( V_p = V_p(r) \) if \( r \) is distance from the centre of a spherical Earth, and similarly for the other physical properties.

The flat-Earth version has greater importance than the spherical Earth version as a result of the existence of the Earth-flattening transformations. These assert that the mathematics appropriate to Cartesian coordinates can be applied to a geological medium with spherical symmetry (physical properties dependent only on distance from the centre) by prescribed conversion formulae; for illustration, velocity in the spherical medium \( V_s \) must be replaced by velocity in the flat medium \( V_f \) by

\[
V_f(r) = (r/r_0) V_s(r).
\]

The method yields an exact transformation for SH waves and an approximate transformation for P–SV waves. Conversion formulae may be found in Aki and Richards (1980).

This class of models has much to recommend it and is widely used. It provides a good approximation to the geology under the common expectation that physical properties tend to be depth dependent, related to effects of temperature or pressure. It also produces simpler mathematics than subsequent classes of models. Most synthetic seismogram computations have been applied to models of this class, in which the physical properties depend upon only one coordinate.

A subclass includes models where the depth (or radial) variation takes the form of uniform layers separated by sharp interfaces. Much of petroleum exploration and most of engineering seismology is based upon this model or perturbations of it. The mathematics is particularly simple. In regions of flat lying or gently dipping sedimentary rocks, the model may be a good approximation to the geology.

Despite the obvious advantages of the depth dependent class of models, they fail to approximate the geology in many important practical cases. These
excluded cases include strongly dipping interfaces in petroleum exploration, the ocean–continent transition in crustal exploration, and most ore bodies in mining exploration. Some of these can be handled by perturbation techniques, but most cannot.

The next step in generalising the model introduces two-dimensional space-dependence for the physical properties. This may provide a suitable approximation for elongated structures such as anticlines. The final generalisation would be to full three-dimensional physical property dependence.

Each increase in complexity of the model reduces the number of computation methods capable of yielding solutions, at least within reasonable computing times. The more complex models can only be handled by the more elementary methods like Asymptotic Ray Theory or finite difference calculations.

**Specification of the Model. III: The Source**

The source creates the seismic signal. Examples include an earthquake, an explosion and a weight drop. For purposes of computing synthetic seismograms, one can describe the source in terms of mechanical models. These may take the form of time-dependent displacements, forces or stresses, specified over or within a small volume surrounding the source. Examples are shown in Fig. 2. Such models can be made more realistic (but correspondingly more complicated) by permitting both time dependence and space dependence, as for example with a propagating fault (Cullen and Douglas 1975). Problems of this type can usually be reduced to the previous format by treating them as the summation of small elements (Green’s function approach).

An alternative approach for synthetic seismograms takes the form of stipulating the amplitude for P waves and/or S waves along a hypothetical small spherical surface surrounding the source as a function of departure angle (Fig. 3). For some sources, the signal may be independent of azimuth (φ) hence only dependence upon θ needs to be specified.

Whichever approach is taken, the required information about the source is the following:

1. Location.

   - geometry of the displacement, force or stress,
   - amplitude of P and/or S motion as a function of departure direction from the source, at an arbitrary small distance from the source.

2. Time history of the source waveform.

   The time history may be based upon observational evidence such as measurements close to the source or within a nearby drill hole. A waveform which has sometimes been used in petroleum exploration (O’Brien 1970) includes one large peak followed by one or two small overshoots (Fig. 4).

**Specification of the Model. IV: The Receiver**

The receiver system contains three components (Fig. 5). In general, all three components are frequency-dependent, hence capable of producing amplitude and phase distortion, i.e. frequency-dependent time delay of
the signal. The mechanical ground motion includes both the free surface effect (if the transducer is at the free surface) and mechanical interaction due to coupling of the transducer to the ground. The problem is further complicated because the transducer may produce an output voltage or current which is proportional to

- ground particle displacement (vertical or horizontal)
- ground particle velocity (vertical or horizontal)
- ground particle acceleration (vertical or horizontal)
- ground strain
- pressure

or some combination of these.

All of these matters must be taken into account, but the final result of such analysis may be compressed into some single statement as, for illustration, the following:

\[ R(w) = \frac{\text{trace displacement amplitude}}{\text{vertical component of ground displacement}} \]

where \( R(w) \) is the transfer function of the receiving system in the frequency domain [alternatively, it transforms \( r(t) \) is the impulse response in the time domain].

**Specification of the Model. V: An Example**

We may illustrate the preceding by presenting the parameters for a model which would be sufficient to permit synthetic seismogram computations. This represents only one possibility from an unlimited number.

1. Free surface = plane.
2. Physical property variation depends only upon the single space variable, depth.
3. Physical properties are specified at discrete depth points—compressional velocity, for example (Fig. 6).

(4) Between data points, depth dependence of physical properties is taken to be linear.
(5) No anelastic attenuation exists.
(6) A point explosive source exists at depth \( Z_s \). The source produces only compressional waves spreading uniformly in all directions. Amplitude of the waves = 1.0 at a distance from the source of one unit. Time history of the source waveform is an impulse.
(7) A displacement receiver exists at the free surface, at a horizontal offset distance \( X \). It responds to vertical ground motion, with system frequency response \( R(w) \).

**Synthetic Seismograms by Asymptotic Ray Theory**

In Asymptotic Ray Theory (ART), also referred to as Geometrical Ray Theory, we are concerned with all of the standard concepts and relationships of classical ray theory. These include rays, wavefronts, Snell's Law, etc. Our intent here is twofold: to call attention to the limitations and restrictions associated with ART and to show how synthetic seismograms can be constructed if such restrictions are acceptable.

We should emphasise that ART is a useful, often an indispensable, basis for all methods of synthetic seismogram computation. Other methods may relax the assumptions implicit in ART and thereby achieve more general results, but the first approximation through ART can give starting values, order-of-magnitude estimates, checks and useful physical insights. Thus, the present section should not be viewed as being mutually exclusive with following sections.

ART is usually presented mathematically as a high-frequency approximation or limit. This can be demonstrated by representing a solution to the elastic equations as an infinite power series in inverse frequency. ART emerges by taking only the first term in the series.

The assumptions and limitations which must be made and accepted in order for ART to be valid are as follows:

1. Linear stress–strain relationship. This assumption is implicit in nearly all methods for computing synthetic seismograms.
2. The geological medium can be subdivided into subvolumes with uniform physical properties. Under this restriction, the classical ray theory (ART) solutions will be exact within the interior of each subvolume. In many practical cases, however, the assumption is relaxed to permit gradual variations of the physical properties across the submedium. Solutions to the elastic equations then become only approximate. The approximation will usually be satisfactory provided the following condition is satisfied:
   - the fractional change in physical properties must be small over a distance of one wavelength of the signal.
3. The boundaries between subvolumes must have negligible thickness as a fraction of a wavelength.
(4) The radii of curvature of the boundaries must either be very large or (locally) very small, expressed as a fraction of a wavelength.

Within these assumptions and approximations, we then emerge with the familiar concepts of classical ray theory. We introduce one amplitude effect, geometrical spreading, based on the following argument (Fig. 7).

Fig. 7. Geometrical spreading of a ray bundle.

Neglecting attenuation (which can be introduced separately if desired), if we consider any bundle of rays anywhere in the medium, the total seismic energy flux passing through any cross-sectional area per unit of time must be constant. This statement derives from the conservation of energy. If we represent the energy per unit volume as being equal to the peak kinetic energy, 

\[(\text{density}) \times (\text{particle velocity})^2\]

the above statement reduces to a requirement for constancy of

\[A^2 \times \text{density} \times \text{cross-sectional-area} \times \text{propagation velocity}\]

at all points along the ray path, where \(A\) represents the displacement amplitude for P waves or for S waves.

The computation of synthetic seismograms within the limits of ART takes the form of computing —travel times along ray paths between source and receiver, —relative amplitudes.

The following factors can be incorporated into the amplitude computations. It should be emphasised that many of these factors can be incorporated into any method of synthetic seismogram computations, so that their importance to synthetic seismograms is more general than for ART computations.

Source effects
(1) Total energy or normalised amplitude.
(2) Relative generation of P versus S waves.
(3) Radiation pattern: relative amplitude as a function of departure angle from the source (vertical and horizontal angles).
(4) Source waveform.

Receiver effects
(1) Sensitivity.
(2) Directionality.
(3) Frequency response or impulse response.
(4) Coupling to medium.
(5) Free surface effect if receiver is on free surface.

Propagation effects
(1) Geometrical spreading, i.e. convergence or divergence of ray paths.
(2) Reflection and transmission effects at boundaries.
(3) Mode conversions at boundaries (P to S, or S to P).
(4) Multiple reflections between boundaries.
(5) Frequency-dependent attenuation along the path.

It should be noted that the only frequency-dependent factors after the signal has left the source are receiver response and (if it is included) attenuation. Thus, the source waveform will propagate without change of waveform except for these factors (note that there can also be reflection and transmission factors such as for wide-angle reflections where a phase shift can occur).

The computation of amplitudes on the basis of classical ray theory has been treated by many authors. For a spherically layered Earth, the method presented by Bullen (1961) is particularly elegant. He shows that if the velocity variation between successive interfaces can be approximated by a power law,

\[V = AR^n\]

then the integration of the differential equations can be carried out analytically, thus reducing the computation to a simple summation. His method does not yield synthetic seismograms as such, however, but simply the amplitude factors. Furthermore, this method, as well as other ART methods, yields amplitude singularities (infinities) associated with every depth point in the medium where either the velocity or the velocity gradient is discontinuous. Various ad hoc arrangements have been proposed to overcome this problem, reviewed by Green (1976), but none is entirely satisfactory.

The ART method has some major advantages. The computations are often simple, straightforward and easy to understand in terms of the underlying physics. The synthetics may be computed to include as many or as few contributing effects (geometrical spreading, attenuation, etc) as seem relevant to the problem at hand. Very few other methods are capable of treating geological models whose physical properties vary horizontally as well as vertically. Examples of such models include economically important structures like ore bodies, salt domes and reefs.

The ART method has some major limitations, however. These include:

(1) The ray paths must be specified in advance. If multiple reflections are present, for example, the number of possible ray paths can become very large.
(2) In non-uniform media, the solutions are only approximate. The degree of approximation may become quite severe.
(3) The computation of amplitudes does not translate directly into a seismic waveform as required for synthetic seismograms. The amplitude variation is defined...
only as a function of distance, not of distance and time, so it cannot be used to construct synthetic seismograms. In addition, phase effects can produce dramatic changes in the waveform.

(4) Geometrical Ray Theory breaks down in many circumstances of seismic interest. Frequency-dependent effects appear which ray theory cannot quantify. For example (Aki and Richards 1980):

—Body waves are seen in shadow zones, where ray theory says they should not be.
—Body waves are seen near caustics (a surface that is a weak type of focus, on which the geometrical spreading factor is singular).
—Complicated interference effects may appear where a variety of rays have similar arrival times.
—Ray theory does not handle diffractions well.

(5) ART does not handle head waves well except for determining arrival times. Amplitude computations are difficult and somewhat ad hoc.

(6) Attenuation and anelastic phenomena must be inserted by some ad hoc arrangement.
(7) ART does not deal satisfactorily with surface waves.

Anelastic attenuation can be introduced into the calculations in either the time or frequency domain. In the time domain, the computed synthetic seismogram can be convolved with an ‘attenuation operator’. A convenient and practical operator, satisfying causality conditions, is given by Carpenter (1966). The frequency domain equivalent is given by Fitch, McCowan and Shields (1980).

Asymptotic Ray Theory has an extensive literature, notably a monograph by Cerveny, Molotkov and Psencik (1977). An example of application is given by McMechan and Mooney (1980). An example of the bookkeeping problems arising from multiple reflections in petroleum exploration is given by May and Hron (1978). A clear explanation of how to incorporate the various effects into a synthetic seismogram is given by Douglas, Hudson and Blamey (1972). Geometrical spreading computations in laterally inhomogeneous media are described by Cerveny and Psencik (1979).

One special version of the ART approach has great importance in petroleum exploration. This involves the one-dimensional problem of plane waves propagating vertically through a horizontally layered medium. The importance derives from the fact that velocity well log data make it possible to construct realistic geologic models with up to hundreds of layers. The number of possible multiple reflections in such a medium defies comprehension.

Computations for this model focus on reflection-transmission coefficients at the interfaces, phase shifts across each layer and anelastic attenuation if desired. The synthetic seismograms can be computed using a matrix formulation described by Claerbout (1968, 1976). A useful artifice to speed computation is to break the model layering into units with equal two-way travel times (Goupillaud 1961). The computations are simplified because, under the assumed conditions, no mode conversion takes place between P and S wave. A formulation convenient for computation is given by Temme and Müller (1982).

In use, these synthetic seismograms can display those seismic arrivals which will have significant amplitudes. Since the ray path for each arrival on the synthetic can usually be assigned, comparison with the field seismogram can provide identification for the observed reflections. Figure 8 shows an example using two modes of display for the synthetic. The reflection coefficients were taken from the sonic log, with the densities inferred using a velocity–density relationship.

Synthetic seismograms based on these one-dimensional models offer hope for solving a long-standing problem in seismic exploration: does the observed loss of high frequencies in the reflected pulse arise from anelastic attenuation or from interference?
effects in finely layered sediments? Recent attempts to resolve the matter using synthetic seismograms are given by Spencer, Sonnad and Butler (1982) and Richards and Menke (1983).

Synthetic Seismograms by Finite Difference Methods
We turn next to more powerful computation methods which can overcome the limitations inherent in ART. Specifically, we hope to treat some of the following: head waves, diffractions, caustics, turning rays and non-ray-path energy. Further, we seek methods which handle these and other effects automatically, thus avoiding the ad hoc quality of ART computations.

The most powerful method currently available uses a numerical approach based upon finite difference computations. The geological medium is (conceptually) broken into tiny volume elements. The differential equations which govern seismic wave propagation are approximated by discrete difference equations.

These methods are straightforward in concept and flexible in execution. They can be applied to geological models of considerable complexity, including models with two- and even three-dimensional velocity variations. They yield very complete seismograms including head waves, diffractions and surface waves.

The most serious problem with finite difference methods lies in computing costs. These become serious for two-dimensional models and often prohibitive for three-dimensional models. The difficulty is worsened by the artificial boundaries resulting from limited computer memory, requiring either a large model or sophisticated absorbing boundary conditions. As a result, the computer algorithm must be ingeniously crafted. Another difficulty arises from the very complexity of the computed synthetics, which may be as difficult to interpret as real data. The solutions also do not easily generalise to a larger class of problems.

Finite difference synthetic seismograms are given, for example, by Kelly, Ward and Treitel (1976). Formulations of the equations in a form suitable for computer programming may be found in Stephen (1983) and Hatherly (1983). Similar computations based upon the finite element method are given by Smith (1975).

Synthetic Seismograms by Transform Methods
Beyond ART and finite differences, most of the remaining techniques are based on transform methods. They differ from one another principally in procedures to take the inverse transform. Nearly all of them are limited to models in which velocity depends only upon depth. The common starting point for these methods is a second-order differential equation in the form

$$\nabla^2 u = \frac{1}{V(x,y,z)^2} \frac{\delta^2 u}{\delta t^2}.$$  

The dependent variable $u$ will typically represent a displacement potential or a component of the displacement vector. We seek a solution which provides a higher degree of approximation than the ART methods, although usually within the requirement that $V(x,y,z)$ be 'slowly varying'. The mathematical technique for this approximation is known as the WKBJ (Wentzel–Kramers–Brillouin–Jeffreys) method.

With $V = V(z)$ only, we take transforms with respect to time and to horizontal distance. The equation then reduces to an ordinary differential equation in $z$ which can be solved by standard techniques. The boundary and source conditions can also be transformed and used to assign the constants of integration. The final result of this process will be the doubly transformed quantity, $\tilde{u}(\rho,z,w)$. The transform to be used depends upon the range of integration for the variable. A Fourier transform can be used for time, which extends from minus to plus infinity. A Fourier–Bessel transform would be required for radial distance, which extends from zero to infinity.

The choice of coordinate system depends upon the geometries of the source and the Earth model. Cartesian coordinates are appropriate for a line source, or cylindrical coordinates for a point source. If the point source has no azimuthal dependence (an explosion, for example), a single solution will suffice. If the point source has azimuthal dependence (fault motion, for example), the synthetic seismograms will differ for different azimuths. Spherical coordinates will be required for a spherical Earth model. The final solution is then obtained as formal expressions for the synthetic seismograms, requiring two inverse transforms, the precise form of the expressions being a function of the coordinate system we are using.

At this point, the synthetic seismogram methods diverge, depending upon how the two integrations are carried out. Chapman (1978) has grouped them in the following way. 'Spectral' methods are those in which the first integration is carried out over $\rho$, the wave slowness, followed by integration over frequency. 'Slowness' methods integrate first over frequency, then over wave slowness. A further characterisation can be made, depending upon whether the slowness integration is carried out along the real axis or along a complex contour.

Each of the four possibilities leads to a specific synthetic seismogram method.

<table>
<thead>
<tr>
<th>Spectral methods</th>
<th>Real slowness contour</th>
<th>Complex slowness contour</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reflectivity method</td>
<td>Full wave theory</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Slowness methods</th>
<th>WKBJ method</th>
<th>Generalised ray theory</th>
</tr>
</thead>
</table>

It is customary to refer to the spectral group as frequency domain methods and the slowness group as time domain methods. We will now review the techniques briefly (a more detailed survey is given in Kennett 1983).
Reflectivity

The reflectivity method has been most widely used amongst the transform techniques. The basic paper is by Fuchs and Müller (1971). The Earth model is taken to be a stack of uniform horizontal layers underlain by a half-space. In the original formulation, both source and receiver were at the free surface. The essence of the method is to separate the Earth model into two separate stacks and to apply different mathematical techniques to the two.

In the upper region, we consider spherical wavefronts emanating from a point source. At interfaces, we use generalised reflection and transmission coefficients (Spencer 1960) appropriate to curved wavefronts. Originally no multiple reflections, free surface reflections nor mode conversions were permitted. In the lower region, the simpler expressions for plane wave reflection and transmission coefficients are used, but multiple reflections and mode conversions are permitted without limit. The integration over wave slowness is carried out numerically by trapezoidal rule or equivalent. Fuchs and Müller chose to convert the integration variable to ray angle, though now slowness is more frequently used.

Reflectivity offers a straightforward and accurate method to compute complete synthetic seismograms. By restricting the integration to certain ranges of ray angle, one can select or eliminate desired seismic arrivals such as head waves. Internal multiples and P-to-S conversions are handled automatically. The most serious difficulty lies in computation costs, especially with a model consisting of many layers. The subroutine which computes reflection and transmission coefficients must be called hundreds or thousands of times during the numerical integration.

In recognition of the basic power of the method, many improvements and extensions have been made. Computational efficiency was increased by a faster subroutine for calculating the reflectivity coefficients (Kind 1976) and also in a series of papers by Kennett and others (Kennett 1980; Kennett and Illingworth 1981; Kennett and Clarke 1983). Extensions to a source at depth and to single force and dislocation-type sources were made by Kind (1978, 1979) and Kennett (1980). Difficulties arising from numerical instability at high frequencies were overcome by Kennett and Kerry (1979). Fryer (1980) converted to a slowness method to make the intermediate results simpler to interpret.

Anelastic attenuation was introduced by Kennett (1975) by replacing the real velocity parameter with a complex velocity independent of frequency,

\[ V = V_R + iV_I \]

with only slight increase in computing costs. Computer programs have been released for public use by Kennett, Kind and Braile.

Figure 9 shows an example of synthetic seismograms computed by the reflectivity method for the velocity-depth model shown in the inset. With sufficient computational resources complete theoretical seismograms including body waves, surface waves and all free surface reflections can be calculated by this approach.

Generalised Ray Theory (GRT)

This method is based upon an ingenious technique to evaluate the inverse transform, originally presented by Cagniard (1939) and improved by de Hoop (1960). By deforming the contour of integration into a carefully chosen path in the complex plane, it becomes possible to reduce the result to a recognisable integral and to avoid actually performing either integration. The method was extended to an explosive source in a layered medium by Helmberger (1968) and has subsequently been used extensively for studying upper Mantle structure (for example, Helmberger and Wiggins 1971).

The method works best for a short time window, in a model with few layers. For a broad-band pulse whose amplitude is changing rapidly with time, details of this change can be easily sampled along the path. Computation costs are modest. The problems with the method arise from two causes. First, the ray paths which will contribute to the signal must be selected in advance. The ray set must be large enough for accuracy yet small enough to be practical. This can present difficulties in shadow zones or where multiple reflections are important. Hron (1971) offers guidance for the selection process. Secondly, programming tends to be difficult because the complex contour must be found iteratively and the solution includes singularities, although Helmberger has developed fast programs with many approximations. Anelastic attenuation can be incorporated by applying an attenuation operator to the synthetic seismograms as with ART; a version of this which satisfies causality is given by Fitch, McCowan and Shields (1980).

Figure 10 shows synthetic seismograms computed using GRT. The inferred earthquake focal mechanism and the locations of the observing stations are shown on...
an equal-area projection. The observed waveforms (dark lines) at 10 stations are compared with synthetics for an 8-km focal depth.

Three approximations have been made to the GRT method. Wiggins and Madrid (1974) have obtained an empirical approximation which they called 'quantised ray theory'. It is rapid and remains valid at caustics. Since it is empirical and without theoretical justification, however, its range of applicability is difficult to determine.

More recently, Wiggins (1976) has obtained by an intuitive physical argument another approximation which he called 'disc ray theory'. Chapman (1976) has arrived at the same result mathematically and has shown that its application is limited. Chiang and Braile (in press) applied disc ray theory to geological models with lateral variations.

**WKBJ method**

This method was presented by Chapman (1978) and has been further developed by him (for example, Dey-Sarkar and Chapman 1978). The method is essentially an extension of ART which incorporates non-ray-path energy travelling close to the ray. In this way, the results are equivalent to geometrical ray theory where both techniques are valid, but the WKBJ method remains applicable on direct and reversed branches of travel time curves, at caustics, critical points, and for partial and total reflections from interfaces, head waves and Fresnel diffractions.

The evaluation of the two inversions is achieved using some ingenious rearrangements and approximations which cast the integrals into recognisable forms. As a result, the intermediate expressions are often simple in form and amenable to approximation, to smoothing for stability, and to physical interpretation. Computer time is modest. Like GRT, the method is best adapted to short time windows on the waveform. Similarly, since the ray path must be specified in advance, the method is poorly suited for problems involving many multiple reflections.

Figure 11, computed with the WKBJ method, shows how the reflection waveform character changes in passing through the critical reflection distance. The velocity model (a two-layer crust) is shown in the lower right. The source waveform, a triangular pulse, can be seen at 20 km. The reflection waveform becomes large at the critical distance, develops a negative pulse, and eventually reverses phase entirely. This waveform change would not have been predicted using ART computations.

**Full wave theory**

This method derives from a paper of Scholte (1956) and, following Scholte, has been largely applied to spherical-Earth problems. His contribution was to show how the solution in the frequency domain could be identified with specific seismic ray paths. The method became practical when Phinney and Alexander (1966) and Phinney and Cathles (1969) showed how to evaluate the integral numerically.

The method can handle, in a spherical-Earth model, grazing incidence rays, diffractions, tunnelling and rays
from a cusp or a caustic. It is less useful in the presence of strong velocity gradients or multiple reflections. As in reflectivity, a single integral evaluation can provide results at several distance ranges.

Hybrid Methods; Methods for Laterally Varying Models

A few other methods are available for synthetic seismogram computation which do not fit well into the above classifications. Alekseev and Mikhailenko (1980) carry out the first inverse transform by matrix techniques but revert to finite difference methods for the second. Bouchon (1981) uses a discrete wavenumber representation to evaluate the integral over wavenumber. Cerveny, Popov and Psencik (1982) represent each ray by a beam of finite width rather than a line; since the beam intensity is assumed to be bell-shaped in distribution, they call this a Gaussian beam method. Harvey (1981) uses a superposition method related to reflectivity. Chapman and Drummond (1982) describe a method which avoids the usual problems at caustics.

For media with lateral as well as vertical changes in velocity, the range of available methods drops drastically. As noted above, ART and finite difference methods will be the usual methods of choice. Hong and Helmerger (1978) used a ray theory method which they called 'Glorified Optics' to treat models with non-planar interfaces. Chiang and Braile (in press) demonstrate the application of disc ray theory to lateral variations. Kennett (1975) extended reflectivity to permit different velocity structures beneath source and receiver. Frazer and Phinney (1980) consider smooth lateral variations without velocity discontinuities.

References

Goupillaud, P. 1961. An approach to inverse filtering of near surface layer effects from seismic records, Geophysics 26, 754–760.
O'BRIEN, P.N.S. 1970. Some experiments concerning the primary seismic pulse, Geophysical Prospecting 17, 511–547.